CHAPTER 3

NETWORK ADMITTANCE AND IMPEDANCE MATRICES

As we have seen in Chapter 1 that a power system network can be converted into an equivalent impedance diagram. This diagram forms the basis of power flow (or load flow) studies and short circuit analysis. In this chapter we shall discuss the formation of bus admittance matrix (also known as \( Y_{bus} \) matrix) and bus impedance matrix (also known as \( Z_{bus} \) matrix). These two matrices are related by

\[
Z_{bus} = Y_{bus}^{-1}
\]

We shall discuss the formation of the \( Y_{bus} \) matrix first. This will be followed by the discussion of the formation of the \( Z_{bus} \) matrix.

3.1 FORMATION OF BUS ADMITTANCE MATRIX

Consider the voltage source \( V_S \) with a source (series) impedance of \( Z_S \) as shown in Fig. 3.1 (a). Using Norton’s theorem this circuit can be replaced by a current source \( I_S \) with a parallel admittance of \( Y_S \) as shown in Fig. 3.1 (b). The relations between the original system and the Norton equivalent are

\[
I_S = \frac{V_S}{Z_S} \quad \text{and} \quad Y_S = \frac{1}{Z_S}
\]

We shall use this Norton’s theorem for the formulation of the \( Y_{bus} \) matrix.

For the time being we shall assume the short line approximation for the formulation of the bus admittance matrix. We shall thereafter relax this assumption and use the \( \pi \)-representation of the network for power flow studies. Consider the 4-bus power system shown in Fig. 3.2. This contains two generators \( G_1 \) and \( G_2 \) that are connected through transformers \( T_1 \) and \( T_2 \) to buses 1 and 2. Let us denote the synchronous reactances of \( G_1 \) and \( G_2 \) by \( X_{G1} \) and \( X_{G2} \) respectively and the leakage reactances of \( T_1 \) and \( T_2 \) by \( X_{T1} \) and \( X_{T2} \) respectively. Let \( Z_{ij} \), \( i = 1, \ldots, 4 \) and \( j = 1, \ldots, 4 \) denote the line impedance between buses \( i \) and \( j \).
Then the system impedance diagram is as shown in Fig. 3.3 where \( Z_{11} = j(X_{G1} + X_{T1}) \) and \( Z_{22} = j(X_{G2} + X_{T2}) \). In this figure the nodes with the node voltages of \( V_1 \) to \( V_4 \) indicate the buses 1 to 4 respectively. Bus 0 indicates the reference node that is usually the neutral of the Y-connected system. The impedance diagram is converted into an equivalent admittance diagram shown in Fig. 3.4. In this diagram \( Y_{ij} = 1/Z_{ij} \), \( i = 1, \ldots, 4 \) and \( j = 1, \ldots, 4 \). The voltage sources \( E_{G1} \) and \( E_{G2} \) are converted into the equivalent current sources \( I_1 \) and \( I_2 \) respectively using the Norton’s theorem discussed before.
We would like to determine the voltage-current relationships of the network shown in Fig. 3.4. It is to be noted that this relation can be written in terms of the node (bus) voltages $V_1$ to $V_4$ and injected currents $I_1$ and $I_2$ as follows

\[
\begin{bmatrix}
I_1 \\
I_2 \\
0 \\
0
\end{bmatrix} = Y_{bus}
\begin{bmatrix}
V_1 \\
V_2 \\
V_3 \\
V_4
\end{bmatrix}
\] (3.3)

or

\[
\begin{bmatrix}
V_1 \\
V_2 \\
V_3 \\
V_4
\end{bmatrix} = Z_{bus}
\begin{bmatrix}
I_1 \\
I_2 \\
0 \\
0
\end{bmatrix}
\] (3.4)

It can be easily seen that we get (3.1) from (3.3) and (3.4).

Consider node (bus) 1 that is connected to the nodes 2 and 3. Then applying KCL at this node we get

\[
I_1 = Y_{11}V_1 + Y_{12}(V_1 - V_2) + Y_{13}(V_1 - V_3) \\
= (Y_{11} + Y_{12} + Y_{13})V_1 - Y_{12}V_2 - Y_{13}V_3
\] (3.5)

In a similar way application of KCL at nodes 2, 3 and 4 results in the following equations

\[
I_2 = Y_{22}V_2 + Y_{12}(V_2 - V_1) + Y_{23}(V_2 - V_3) + Y_{24}(V_2 - V_4) \\
= -Y_{12}V_1 + (Y_{22} + Y_{12} + Y_{23} + Y_{24})V_2 - Y_{23}V_3 - Y_{24}V_4
\] (3.6)

\[
0 = Y_{13}(V_3 - V_1) + Y_{23}(V_3 - V_2) + Y_{34}(V_3 - V_4) \\
= -Y_{13}V_1 - Y_{23}V_2 + (Y_{13} + Y_{23} + Y_{34})V_3 - Y_{34}V_4
\] (3.7)

\[
0 = Y_{24}(V_4 - V_2) + Y_{34}(V_4 - V_3) \\
= -Y_{24}V_2 - Y_{34}V_3 + (Y_{24} + Y_{34})V_4
\] (3.8)

Combining (3.5) to (3.8) we get

\[
\begin{bmatrix}
I_1 \\
I_2 \\
0 \\
0
\end{bmatrix} =
\begin{bmatrix}
Y_{11} + Y_{12} + Y_{13} & -Y_{12} & -Y_{13} & 0 \\
-Y_{12} & Y_{22} + Y_{12} + Y_{23} + Y_{24} & -Y_{23} & -Y_{24} \\
-Y_{13} & -Y_{23} & Y_{13} + Y_{23} + Y_{34} & -Y_{34} \\
0 & 0 & -Y_{24} & -Y_{34}
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2 \\
V_3 \\
V_4
\end{bmatrix}
\] (3.9)
Comparing (3.9) with (3.3) we can write

\[ Y_{bus} = \begin{bmatrix}
Y_{11} + Y_{12} + Y_{13} & -Y_{12} & -Y_{13} & 0 \\
-Y_{12} & -Y_{22} + Y_{12} + Y_{24} & -Y_{23} & -Y_{24} \\
-Y_{13} & -Y_{23} & Y_{23} + Y_{34} & -Y_{34} \\
0 & -Y_{24} & -Y_{34} & Y_{24} + Y_{34}
\end{bmatrix} \]  

(3.10)

In general the format of the \( Y_{bus} \) matrix for an \( n \)-bus power system is as follows

\[ Y_{bus} = \begin{bmatrix}
y_1 & -y_{12} & -y_{13} & \cdots & -y_{1n} \\
-y_{12} & y_2 & -y_{23} & \cdots & -y_{2n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-y_{1n} & -y_{2n} & -y_{3n} & \cdots & y_n
\end{bmatrix} \]  

(3.11)

where

\[ y_k = \sum_{j=1}^{n} y_{kj} \]  

(3.12)

It is to be noted that \( Y_{bus} \) is a symmetric matrix in which the sum of all the elements of the \( k \)th column is \( y_{kk} \).

**Example 3.1:** Consider the impedance diagram of Fig. 3.2 in which the system parameters are given in per unit by

\[ Z_{11} = Z_{22} = j0.25, Z_{12} = j0.2, Z_{13} = j0.25, Z_{23} = Z_{34} = j0.4 \text{ and } Z_{24} = j0.5 \]

The system admittance can then be written in per unit as

\[ y_{11} = y_{22} = -j4, y_{12} = -j5, y_{13} = -j4, y_{23} = y_{34} = -j2.5 \text{ and } y_{24} = -j2 \]

The \( Y_{bus} \) is then given from (3.10) as

\[ Y_{bus} = \begin{bmatrix}
-13 & 5 & 4 & 0 \\
5 & -13.5 & 2.5 & 2 \\
4 & 2.5 & -9 & 2.5 \\
0 & 2 & 2.5 & -4.5
\end{bmatrix} \text{ per unit} \]

Consequently the bus impedance matrix is given by

\[ Z_{bus} = \begin{bmatrix}
0.1531 & 0.0969 & 0.1264 & 0.1133 \\
0.0969 & 0.1531 & 0.1236 & 0.1367 \\
0.1264 & 0.1236 & 0.2565 & 0.1974 \\
0.1133 & 0.1367 & 0.1974 & 0.3926
\end{bmatrix} \text{ per unit} \]
It can be seen that like the $Y_{bus}$ matrix the $Z_{bus}$ matrix is also symmetric. Let us now assume that the voltages $E_{G1}$ and $E_{G2}$ are given by

$$E_{G1} = 1\angle 30^\circ \text{ pu} \quad \text{and} \quad E_{G2} = 1\angle 0^\circ \text{ pu}$$

The current sources $I_1$ and $I_2$ are then given by

$$I_1 = \frac{E_{G1}}{Z_{11}} = \frac{1\angle 30^\circ}{0.25\angle 90^\circ} = 4\angle -60^\circ \text{ pu}$$

$$I_2 = \frac{E_{G2}}{Z_{22}} = \frac{1\angle 0^\circ}{0.25\angle 90^\circ} = 4\angle -90^\circ \text{ pu}$$

We then get the node voltages from (3.4) as

$$
\begin{bmatrix}
V_1 \\
V_2 \\
V_3 \\
V_4
\end{bmatrix}
= 
\begin{bmatrix}
0.1531 & j0.0969 & j0.1264 & j0.1133 \\
0.0969 & j0.1531 & j0.1236 & j0.1367 \\
j0.1264 & j0.1236 & j0.2565 & j0.1974 \\
j0.1133 & j0.1367 & j0.1974 & j0.3926
\end{bmatrix}
\begin{bmatrix}
4\angle -60^\circ \\
4\angle -90^\circ \\
0 \\
0
\end{bmatrix}
$$

Solving the above equation we get the node voltages as

$$
\begin{bmatrix}
V_1 \\
V_2 \\
V_3 \\
V_4
\end{bmatrix}
= 
\begin{bmatrix}
0.9677\angle 18.45^\circ \\
0.9677\angle 11.55^\circ \\
0.9659\angle 15.18^\circ \\
0.9662\angle 13.56^\circ
\end{bmatrix} \text{ per unit}
$$

### 3.1.1 Node Elimination by Matrix Partitioning

Sometimes it is desirable to reduce the network by eliminating the nodes in which the current do not enter or leave. Let (3.3) be written as

$$
\begin{bmatrix}
I_A \\
I_s
\end{bmatrix}
= 
\begin{bmatrix}
K & L \\
L^T & M
\end{bmatrix}
\begin{bmatrix}
V_A \\
V_s
\end{bmatrix}
$$

In the above equation $I_A$ is a vector containing the currents that are injected, $I_s$ is a null vector and the $Y_{bus}$ matrix is portioned with the matrices $K$, $L$ and $M$. Note that the $Y_{bus}$ matrix contains both $L$ and $L^T$ due to its symmetric nature.

We get the following two sets of equations from (3.13)

$$I_A = KV_A + LV_s \quad \Rightarrow \quad I_A = LV_s$$

$$0 = I_s = L^TV_A + MV_s \quad \Rightarrow \quad V_s = -M^{-1}L^TV_A$$
Substituting (3.15) in (3.14) we get

\[ I_A = (K - LM^{-1}L^T)V_A \]

Therefore we obtain the following reduced bus admittance matrix

\[ Y_{bus}^{\text{reduced}} = K - LM^{-1}L^T \]  

(3.17)

**Example 3.2**: Let us consider the system of Example 3.1. Since there is no current injection in either bus 3 or bus 4, from the \( Y_{bus} \) computed we can write

\[ K = \begin{bmatrix} -j13 & j5 \\ j5 & -j13.5 \end{bmatrix}, \quad L = \begin{bmatrix} j4 & 0 \\ j2.5 & j2 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} -j9 & j2.5 \\ j2.5 & -j4.5 \end{bmatrix} \]

We then have

\[ Y_{bus}^{\text{reduced}} = K - LM^{-1}L^T = \begin{bmatrix} -j10.8978 & j6.8978 \\ j6.8978 & -j10.8978 \end{bmatrix} \text{ per unit} \]

Substituting \( I_1 = 4 \angle -60^\circ \) per unit and \( I_2 = 4 \angle -90^\circ \) per unit we shall get the same values of \( V_1 \) and \( V_2 \) as given in Example 3.1.

Inspecting the reduced \( Y_{bus} \) matrix we can state that the admittance between buses 1 and 2 is \( -j6.8978 \). Therefore the self admittance (the admittance that is connected in shunt) of the buses 1 and 2 is \( -j4 \) per unit (= \( -j10.8978 + j6.8978 \)). The reduced admittance diagram obtained by eliminating nodes 3 and 4 is shown in Fig. 3.5. It is to be noted that the impedance between buses 1 and 2 is the Thevenin impedance between these two buses. The value of this impedance is \( 1/(−j6.8978) = j0.145 \) per unit.

![Fig. 3.5 Reduced admittance diagram after the elimination of buses 3 and 4.](image)

### 3.1.2 Node Elimination by Kron Reduction

Consider an equation of the form

\[ Ax = b \]

(3.18)

where \( A \) is an \((n \times n)\) real or complex valued matrix, \( x \) and \( b \) are vectors in either \( \mathbb{R}^n \) or \( \mathbb{C}^n \). Assume that the \( b \) vector has a zero element in the \( n \)th row such that (3.18) is given as
We can then eliminate the \( k \)th row and \( k \)th column to obtain a reduced \((n - 1)\) number of equations of the form

\[
\begin{bmatrix}
    a_{11} & a_{12} & \cdots & a_{1,n-1} \\
    a_{21} & a_{22} & \cdots & a_{2,n-1} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1}
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2 \\
    \vdots \\
    x_{n-1}
\end{bmatrix}
= 
\begin{bmatrix}
    b_1 \\
    b_2 \\
    \vdots \\
    b_{n-1}
\end{bmatrix}
\]

(3.19)

The elimination is performed using the following elementary operations

\[ a_{nj}^{\text{new}} = \frac{a_{kj}a_{nj}}{a_{nn}} \]  

(3.21)

\textbf{Example 3.3}: Let us consider the same system of Example 3.1. We would like to eliminate the last two rows and columns. Let us first eliminate the last row and last column. Some of the values are given below

\[
\begin{align*}
Y_{21}^{\text{new}} &= Y_{21} - \frac{Y_{22}Y_{41}}{Y_{44}} = j5, \\
Y_{22}^{\text{new}} &= Y_{22} - \frac{Y_{24}Y_{42}}{Y_{44}} = -j13.5 + j2 \times \frac{2}{4.5} = -j12.6111 \\
Y_{23}^{\text{new}} &= Y_{23} - \frac{Y_{24}Y_{43}}{Y_{44}} = j2.5 + j2 \times \frac{2.5}{4.5} = j3.6111, \\
Y_{4}^{\text{new}} &= Y_{44} - \frac{Y_{44}}{Y_{44}} = 0
\end{align*}
\]

In a similar way we can calculate the other elements. Finally eliminating the last row and last column, as all these elements are zero, we get the new \( Y_{\text{bus}} \) matrix as

\[
Y_{\text{bus}}^{\text{new}} = \begin{bmatrix}
    -j13 & j5 & j4 \\
    j5 & -j12.6111 & j3.6111 \\
    j4 & j3.6111 & -j7.6111
\end{bmatrix}
\]

Further reducing the last row and the last column of the above matrix using (3.21), we obtain the reduced \( Y_{\text{bus}} \) matrix given in Example 3.2.

2.1.3 Inclusion of Line Charging Capacitors

So far we have assumed that the transmission lines are modeled with lumped series impedances without the shunt capacitances. However in practice, the \( Y_{\text{bus}} \) matrix contains the shunt admittances for load flow analysis in which the transmission lines are represented by its
\( \pi \)-equivalent. Note that whether the line is assumed to be of medium length or long length is irrelevant as we have seen in Chapter 2 how both of them can be represented in a \( \pi \)-equivalent.

Consider now the power system of Fig. 3.2. Let us assume that all the lines are represented in an equivalent-\( \pi \) with the shunt admittance between the line \( i \) and \( j \) being denoted by \( Y_{chij} \). Then the equivalent admittance at the two end of this line will be \( Y_{chij}/2 \). For example the shunt capacitance at the two ends of the line joining buses 1 and 3 will be \( Y_{ch13}/2 \). We can then modify the admittance diagram Fig. 3.4 as shown in Fig. 3.6. The \( Y_{bus} \) matrix of (3.10) is then modified as

\[
Y_{bus} = \begin{bmatrix}
Y_{11} + Y_{12} + Y_{13} + Y_{ch1} & -Y_{12} & -Y_{13} & 0 \\
-Y_{12} & Y_{12} + Y_{13} + Y_{24} + Y_{ch2} & -Y_{13} & -Y_{23} & -Y_{24} \\
-Y_{13} & -Y_{13} & Y_{13} + Y_{23} + Y_{34} + Y_{ch3} & -Y_{34} \\
0 & -Y_{23} & -Y_{24} & -Y_{34} & 0
\end{bmatrix}
\]

(3.22)

where

\[
Y_{ch1} = \frac{Y_{ch12} + Y_{ch13}}{2}
\]

\[
Y_{ch2} = \frac{Y_{ch12} + Y_{ch23} + Y_{ch24}}{2}
\]

\[
Y_{ch3} = \frac{Y_{ch13} + Y_{ch23} + Y_{ch34}}{2}
\]

\[
Y_{ch4} = \frac{Y_{ch24} + Y_{ch34}}{2}
\]

(3.23)

Fig. 3.6 Admittance diagram of the power system Fig. 3.2 with line charging capacitors.
3.2 ELEMENTS OF THE BUS IMPEDANCE AND ADMITTANCE MATRICES

Equation (3.1) indicates that the bus impedance and admittance matrices are inverses of each other. Also since $Y_{bus}$ is a symmetric matrix, $Z_{bus}$ is also a symmetric matrix. Consider a 4-bus system for which the voltage-current relations are given in terms of the $Y_{bus}$ matrix as

$$
\begin{bmatrix}
I_1 \\
I_2 \\
I_3 \\
I_4
\end{bmatrix} =
\begin{bmatrix}
Y_{11} & Y_{12} & Y_{13} & Y_{14} \\
Y_{21} & Y_{22} & Y_{23} & Y_{24} \\
Y_{31} & Y_{32} & Y_{33} & Y_{34} \\
Y_{41} & Y_{42} & Y_{43} & Y_{44}
\end{bmatrix}
\begin{bmatrix}
V_1 \\
V_2 \\
V_3 \\
V_4
\end{bmatrix}
$$

(3.24)

We can then write

$$
Y_{11} = \frac{I_1}{V_1} \bigg|_{V_2=V_3=V_4=0}
$$

(3.25)

This implies that $Y_{11}$ is the admittance measured at bus-1 when buses 2, 3 and 4 are short circuited. The admittance $Y_{11}$ is defined as the *self admittance* at bus-1. In a similar way the self admittances of buses 2, 3 and 4 can also be defined that are the diagonal elements of the $Y_{bus}$ matrix. The off diagonal elements are denoted as the *mutual admittances*. For example the mutual admittance between buses 1 and 2 is defined as

$$
Y_{12} = \frac{I_1}{V_2} \bigg|_{V_1=V_3=V_4=0}
$$

(3.26)

The mutual admittance $Y_{12}$ is obtained as the ratio of the current injected in bus-1 to the voltage of bus-2 when buses 1, 3 and 4 are short circuited. This is obtained by applying a voltage at bus-2 while shorting the other three buses.

The voltage-current relation can be written in terms of the $Z_{bus}$ matrix as

$$
\begin{bmatrix}
V_1 \\
V_2 \\
V_3 \\
V_4
\end{bmatrix} =
\begin{bmatrix}
Z_{11} & Z_{12} & Z_{13} & Z_{14} \\
Z_{21} & Z_{22} & Z_{23} & Z_{24} \\
Z_{31} & Z_{32} & Z_{33} & Z_{34} \\
Z_{41} & Z_{42} & Z_{43} & Z_{44}
\end{bmatrix}
\begin{bmatrix}
I_1 \\
I_2 \\
I_3 \\
I_4
\end{bmatrix}
$$

(3.27)

The *driving point impedance* at bus-1 is then defined as

$$
Z_{11} = \frac{V_1}{I_1} \bigg|_{I_2=I_3=I_4=0}
$$

(3.28)

i.e., the driving point impedance is obtained by injecting a current at bus-1 while keeping buses 2, 3 and 4 open-circuited. Comparing (3.26) and (3.28) we can conclude that $Z_{11}$ is not the reciprocal of $Y_{11}$. The *transfer impedance* between buses 1 and 2 can be obtained by injecting a current at bus-2 while open-circuiting buses 1, 3 and 4 as
\[
Z_{12} = \frac{V_1}{I_2} \bigg|_{I_1=I_3=I_4=0}
\]

(3.29)

It can also be seen that \( Z_{12} \) is not the reciprocal of \( Y_{12} \).

### 3.3 Modification of Bus Impedance Matrix

Equation (3.1) gives the relation between the bus impedance and admittance matrices. However, it may be possible that the topology of the power system changes by the inclusion of a new bus or line. In that case, it is not necessary to recompute the \( Y_{\text{bus}} \) matrix again for the formation of \( Z_{\text{bus}} \) matrix. We shall discuss four possible cases by which an existing bus impedance matrix can be modified.

Let us assume that an \( n \)-bus power system exists in which the voltage-current relations are given in terms of the bus impedance matrix as

\[
\begin{bmatrix}
V_1 \\
v \\
V_n
\end{bmatrix} = Z_{\text{orig}}
\begin{bmatrix}
I_1 \\
v \\
I_n
\end{bmatrix}
\]

(3.30)

The aim is to modify the matrix \( Z_{\text{orig}} \) when a new bus or line is connected to the power system.

#### 3.3.1 Adding a New Bus to the Reference Bus

It is assumed that a new bus \( p \) (\( p > n \)) is added to the reference bus through an impedance \( Z_p \). The schematic diagram for this case is shown in Fig. 3.7. Since this bus is only connected to the reference bus, the voltage-current relations the new system are

\[
\begin{bmatrix}
V_1 \\
v \\
V_n \\
V_p
\end{bmatrix} =
\begin{bmatrix}
Z_{\text{orig}} & 0 & 0 \\
0 & I_1 & \cdots & 0 \\
0 & 0 & \cdots & I_n \\
0 & 0 & \cdots & 0
\end{bmatrix}
\begin{bmatrix}
I_1 \\
v \\
I_n \\
I_p
\end{bmatrix} = Z_{\text{new}}
\]

(3.31)

Fig. 3.7 A new bus is added to the reference bus.
3.3.2 Adding a New Bus to an Existing Bus through an Impedance

This is the case when a bus, which has not been a part of the original network, is added to an existing bus through a transmission line with an impedance of \( Z_b \). Let us assume that \( p \) \((p > n)\) is the new bus that is connected to bus \( k \) \((k < n)\) through \( Z_b \). Then the schematic diagram of the circuit is as shown in Fig. 3.8. Note from this figure that the current \( I_p \) flowing from bus \( p \) will alter the voltage of the bus \( k \). We shall then have

\[
V_k = Z_{k1}I_1 + Z_{k2}I_2 + \cdots + Z_{kk}(I_k + I_p) + \cdots + Z_{kn}I_n
\]

(3.32)

In a similar way the current \( I_p \) will also alter the voltages of all the other buses as

\[
V_i = Z_{i1}I_1 + Z_{i2}I_2 + \cdots + Z_{ik}(I_k + I_p) + \cdots + Z_{in}I_n, \quad i \neq k
\]

(3.33)

Furthermore the voltage of the bus \( p \) is given by

\[
V_p = V_k + Z_b I_p
\]

\[
= Z_{k1}I_1 + Z_{k2}I_2 + \cdots + Z_{kk}I_k + \cdots + Z_{kn}I_n + (Z_{kk} + Z_b)I_p
\]

(3.34)

Therefore the new voltage current relations are

\[
\begin{bmatrix}
V_k \\
\vdots \\
V_p
\end{bmatrix} =
\begin{bmatrix}
Z_{k1} & \cdots & \cdots & \cdots \\\n\vdots & Z_{kk} & \cdots & \cdots \\
\vdots & \vdots & \ddots & \cdots \\
\vdots & \vdots & \cdots & Z_{kn}
\end{bmatrix}
\begin{bmatrix}
I_1 \\
\vdots \\
I_n \\
I_p
\end{bmatrix} = Z_{new}
\begin{bmatrix}
I_1 \\
\vdots \\
I_n \\
I_p
\end{bmatrix}
\]

(3.35)

It can be noticed that the new \( Z_{bus} \) matrix is also symmetric.

![Fig. 3.8 A new bus is added to an existing bus through an impedance.](image)

3.3.3 Adding an Impedance to the Reference Bus from an Existing Bus

To accomplish this we first assume that an impedance \( Z_b \) is added from a new bus \( p \) to an existing bus \( k \). This can be accomplished using the method discussed in Section 3.3.2. Then to add this bus \( k \) to the reference bus through \( Z_b \), we set the voltage \( V_p \) of the new bus to zero. However now we have an \((n + 1) \times (n + 1)\) \( Z_{bus} \) matrix instead of an \( n \times n \) matrix. We can then remove the last row and last column of the new \( Z_{bus} \) matrix using the Kron’s reduction given in (3.21).
3.3.4 Adding an Impedance between two Existing Buses

Let us assume that we add an impedance $Z_b$ between two existing buses $k$ and $j$ as shown in Fig. 3.9. Therefore the current injected into the network from the bus $k$ side will be $I_k - I_b$ instead of $I_k$. Similarly the current injected into the network from the bus $j$ side will be $I_j + I_b$ instead of $I_j$. Consequently the voltage of the $i^{th}$ bus will be

$$V_i = Z_{i1}I_1 + Z_{i2}I_2 + \cdots + Z_{ij}(I_j + I_b) + Z_{ik}(I_k - I_b) + \cdots + Z_{in}I_n$$

$$= Z_{i1}I_1 + Z_{i2}I_2 + \cdots + Z_{ij}I_j + Z_{ik}I_k + \cdots + Z_{in}I_n + (Z_{ij} - Z_{ik})I_b$$

(3.36)

Similarly we have

$$V_j = Z_{ji}I_1 + Z_{j2}I_2 + \cdots + Z_{jj}I_j + Z_{jk}I_k + \cdots + Z_{jn}I_n + (Z_{jj} - Z_{jk})I_b$$

(3.37)

and

$$V_k = Z_{k1}I_1 + Z_{k2}I_2 + \cdots + Z_{kj}I_j + Z_{kk}I_k + \cdots + Z_{kn}I_n + (Z_{kj} - Z_{kk})I_b$$

(3.38)

We can then write the voltage current relations as

\[
\begin{bmatrix}
V_1 \\
\vdots \\
V_n \\
0
\end{bmatrix} =
\begin{bmatrix}
Z_{i1} - Z_{k1} & \cdots & Z_{i1} - Z_{k1} \\
\vdots & \ddots & \vdots \\
Z_{jn} - Z_{kn} & \cdots & Z_{jn} - Z_{kn} \\
0 & \cdots & 0
\end{bmatrix}
\begin{bmatrix}
I_1 \\
\vdots \\
I_n \\
I_p
\end{bmatrix}
\]

\[
= Z_{new}
\begin{bmatrix}
I_1 \\
\vdots \\
I_n \\
I_p
\end{bmatrix}
\]

(3.41)

where
\[ Z_{hh} = Z_h + Z_{ij} - 2Z_{jk} + Z_{kk} \]  

(3.42)

We can now eliminate the last row and last column using the Kron’s reduction given in (3.21).

### 3.3.5 Direct Determination of \( Z_{bus} \) Matrix

We shall now use the methods given in Sections 3.3.1 to 3.3.4 for the direct determination of the \( Z_{bus} \) matrix without forming the \( Y_{bus} \) matrix first. To accomplish this we shall consider the system of Fig. 3.2 and shall use the system data given in Example 3.1. Note that for the construction of the \( Z_{bus} \) matrix we first eliminate all the voltage sources from the system.

**Step-1:** Start with bus-1. Assume that no other buses or lines exist in the system. We add this bus to the reference bus with the impedance of \( j0.25 \) per unit. Then the \( Z_{bus} \) matrix is

\[ Z_{bus,1} = j0.25 \]  

(3.43)

**Step-2:** We now add bus-2 to the reference bus using (3.31). The system impedance diagram is shown in Fig. 3.10. We then can modify (3.43) as

\[
\begin{bmatrix}
    j0.25 & 0 \\
    0 & j0.25
\end{bmatrix}
\]  

(3.44)

**Fig. 3.10 Network of step-2.**

**Step-3:** We now add an impedance of \( j0.2 \) per unit between buses 1 and 2 as shown in Fig. 3.11. The interim \( Z_{bus} \) matrix is then obtained by applying (3.41) on (3.44) as

\[
\begin{bmatrix}
    j0.25 & 0 & j0.25 \\
    0 & j0.25 & -j0.25 \\
    j0.25 & -j0.25 & j0.7
\end{bmatrix}
\]

Eliminating the last row and last column using the Kron’s reduction of (3.31) we get

\[
\begin{bmatrix}
    j0.1607 & j0.0893 \\
    j0.0893 & j0.1607
\end{bmatrix}
\]  

(3.45)

**Step-4:** We now add bus-3 to bus-1 through an impedance of \( j0.25 \) per unit as shown in Fig. 3.12. The application of (3.35) on (3.45) will then result in the following matrix
Step-5: Connect buses 2 and 3 through an impedance of \( j0.4 \) per unit as shown in Fig. 3.13. The interim \( Z_{bus} \) matrix is then formed from (3.41) and (3.46) as

\[
Z_{bus,5}^{in} = \begin{bmatrix}
  j0.1607 & j0.0893 & j0.1607 & -j0.0714 \\
  j0.0893 & j0.1607 & j0.0893 & j0.0714 \\
  j0.1607 & j0.0893 & j0.4107 & -j0.3214 \\
  -j0.0714 & j0.0714 & -j0.3214 & j0.7928 \\
\end{bmatrix}
\]

Using the Kron’s reduction we get the following matrix

\[
Z_{bus,5} = \begin{bmatrix}
  j0.1543 & j0.0957 & j0.1318 \\
  j0.0957 & j0.1543 & j0.1182 \\
  j0.1318 & j0.1182 & j0.2804 \\
\end{bmatrix}
\]

(3.47)
Step-6: We now add a new bus-4 to bus-2 through an impedance of \( j0.5 \) as shown in Fig. 3.14. Then the application of (3.35) on (3.47) results in the following matrix

\[
Z_{bus,6} = \begin{bmatrix}
  j0.1543 & j0.0957 & j0.1318 & j0.0957 \\
  j0.0957 & j0.1543 & j0.1182 & j0.1543 \\
  j0.1318 & j0.1182 & j0.2804 & j0.1182 \\
  j0.0957 & j0.1543 & j0.1182 & j0.6543 \\
\end{bmatrix}
\]

(3.48)

![Fig. 3.14 Network of step-6.](image)

Step-7: Finally we add buses 3 and 4 through an impedance of \( j0.4 \) to obtain the network of Fig. 3.3 minus the voltage sources. The application of (3.41) on (3.48) results in the interim \( Z_{bus} \) matrix of

\[
Z_{bus,7} = \begin{bmatrix}
  j0.1543 & j0.0957 & j0.1318 & j0.0957 & j0.0360 \\
  j0.0957 & j0.1543 & j0.1182 & j0.1543 & -j0.0360 \\
  j0.1318 & j0.1182 & j0.2804 & j0.1182 & j0.1622 \\
  j0.0957 & j0.1543 & j0.1182 & j0.6543 & j0.5360 \\
  j0.0360 & -j0.0360 & j0.1622 & j0.5360 & j1.0982 \\
\end{bmatrix}
\]

Eliminating the 5\(^{th}\) row and column through Kron’s reduction we get the final \( Z_{bus} \) as

\[
Z_{bus,7} = \begin{bmatrix}
  j0.1531 & j0.0969 & j0.1264 & j0.1133 \\
  0.0969 & j0.1531 & j0.1236 & j0.1367 \\
  j0.1264 & j0.1236 & j0.2565 & j0.1974 \\
  j0.1133 & j0.1367 & j0.1974 & j0.3926 \\
\end{bmatrix}
\]

(3.49)

The \( Z_{bus} \) matrix given in (3.49) is the as that given in Example 3.1 which is obtained by inverting the \( Y_{bus} \) matrix.

### 3.4 THEVENIN IMPEDANCE AND \( Z_{bus} \) MATRIX

To establish relationships between the elements of the \( Z_{bus} \) matrix and Thevenin equivalent, let us consider the following example.

**Example 3.4:** Consider the two bus power system shown in Fig. 3.15. It can be seen that the open-circuit voltages of buses \( a \) and \( b \) are \( V_a \) and \( V_b \) respectively. From (3.11) we can write the \( Y_{bus} \) matrix of the system as
Fig. 3.15 Two-bus power system of Example 3.4.

\[
Y_{bus} = \begin{bmatrix}
\frac{1}{Z_{aa}} + \frac{1}{Z_{ab}} & -\frac{1}{Z_{ab}} & -\frac{1}{Z_{ab}} & \frac{1}{Z_{ab}} \\
-\frac{1}{Z_{ab}} & \frac{1}{Z_{ab}} + \frac{1}{Z_{bb}} & \frac{1}{Z_{ab}} & -\frac{1}{Z_{ab}} \\
\frac{1}{Z_{ab}} & \frac{1}{Z_{ab}} & \frac{1}{Z_{ab}} & -\frac{1}{Z_{ab}} \\
-\frac{1}{Z_{ab}} & \frac{1}{Z_{ab}} & \frac{1}{Z_{ab}} & \frac{1}{Z_{ab}} \\
\end{bmatrix}
\]

The determinant of the above matrix is

\[
|Y_{bus}| = \frac{Z_{aa} + Z_{ab} + Z_{bb}}{Z_{aa}Z_{ab}Z_{bb}}
\]

Therefore the \(Z_{bus}\) matrix is

\[
Z_{bus} = Y_{bus}^{-1} = \frac{1}{|Y_{bus}|} \begin{bmatrix}
\frac{1}{Z_{ab}}Z_{bb} & \frac{1}{Z_{ab}} & \frac{1}{Z_{ab}} & \frac{1}{Z_{ab}} \\
\frac{1}{Z_{ab}} & \frac{1}{Z_{ab}}Z_{bb} & \frac{1}{Z_{ab}} & \frac{1}{Z_{ab}} \\
\frac{1}{Z_{ab}} & \frac{1}{Z_{ab}} & \frac{1}{Z_{ab}}Z_{bb} & \frac{1}{Z_{ab}} \\
\frac{1}{Z_{ab}} & \frac{1}{Z_{ab}} & \frac{1}{Z_{ab}} & \frac{1}{Z_{ab}} \\
\end{bmatrix}
\]

Solving the last two equations we get

\[
Z_{bus} = \begin{bmatrix}
Z_{11} & Z_{12} \\
Z_{12} & Z_{22} \\
\end{bmatrix} = \begin{bmatrix}
\frac{Z_{aa}(Z_{ab} + Z_{bb})}{Z_{aa} + Z_{ab} + Z_{bb}} & \frac{Z_{aa}Z_{bb}}{Z_{aa} + Z_{ab} + Z_{bb}} \\
\frac{Z_{aa}Z_{ab}}{Z_{aa} + Z_{ab} + Z_{bb}} & \frac{Z_{bb}(Z_{aa} + Z_{ab})}{Z_{aa} + Z_{ab} + Z_{bb}} \\
\end{bmatrix}
\]

(3.50)

Now consider the system of Fig. 3.15. The Thevenin impedance of looking into the system at bus-\(a\) is the parallel combination of \(Z_{aa}\) and \(Z_{ab} + Z_{bb}\), i.e.,

\[
Z_{th,a} = \frac{Z_{aa}(Z_{ab} + Z_{bb})}{Z_{aa} + Z_{ab} + Z_{bb}} = Z_{11}
\]

(3.51)

Similarly the Thevenin impedance obtained by looking into the system at bus-\(b\) is the parallel combination of \(Z_{bb}\) and \(Z_{aa} + Z_{ab}\), i.e.,

\[
Z_{th,b} = \frac{Z_{bb}(Z_{aa} + Z_{ab})}{Z_{aa} + Z_{ab} + Z_{bb}} = Z_{22}
\]

(3.52)

Hence the driving point impedances of the two buses are their Thevenin impedances.
Let us now consider the Thevenin impedance while looking at the system between the buses \(a\) and \(b\). From Fig. 3.15 it is evident that this Thevenin impedance is the parallel combination of \(Z_{ab}\) and \(Z_{aa} + Z_{bb}\), i.e.,

\[
Z_{th,ab} = \frac{Z_{ab} (Z_{aa} + Z_{bb})}{Z_{aa} + Z_{ab} + Z_{bb}}
\]

With the values given in (3.50) we can write

\[
Z_{11} + Z_{22} - 2Z_{12} = \frac{Z_{aa} (Z_{ab} + Z_{bb})}{Z_{aa} + Z_{ab} + Z_{bb}} + \frac{Z_{bb} (Z_{aa} + Z_{ab})}{Z_{aa} + Z_{ab} + Z_{bb}} - 2 \frac{Z_{aa}Z_{bb}}{Z_{aa} + Z_{ab} + Z_{bb}}
\]

\[
= \frac{1}{Z_{aa} + Z_{ab} + Z_{bb}} [Z_{aa}Z_{ab} + Z_{ab}Z_{bb}]
\]

Comparing the last two equations we can write

\[
Z_{th,ab} = Z_{11} + Z_{22} - 2Z_{12} \quad (3.53)
\]

As we have seen in the above example in the relation \(V = Z_{bus}I\), the node or bus voltages \(V_i\), \(i = 1, \ldots, n\) are the open circuit voltages. Let us assume that the currents injected in buses 1, \(\ldots, k-1\) and \(k+1, \ldots, n\) are zero when a short circuit occurs at bus \(k\). Then Thevenin impedance at bus \(k\) is

\[
Z_{th,k} = \frac{V_k}{I_k} = Z_{kk} \quad (3.54)
\]

From (3.51), (3.52) and (3.54) we can surmise that the driving point impedance at each bus is the Thevenin impedance.

Let us now find the Thevenin impedance between two buses \(j\) and \(k\) of a power system. Let the open circuit voltages be defined by the voltage vector \(V^o\) and corresponding currents be defined by \(I^o\) such that

\[
V^o = Z_{bus}I^o \quad (3.55)
\]

Now suppose the currents are changed by \(\Delta I\) such that the voltages are changed by \(\Delta V\). Then

\[
V = V^o + \Delta V = Z_{bus} (I^o + \Delta I) \quad (3.56)
\]

Comparing (3.55) and (3.56) we can write

\[
\Delta V = Z_{bus} \Delta I \quad (3.57)
\]
Let us now assume that additional currents $\Delta I_k$ and $\Delta I_j$ are injected at the buses $k$ and $j$ respectively while the currents injected at the other buses remain the same. Then from (3.57) we can write

$$
\Delta V = Z_{bus} \begin{bmatrix}
0 \\
\vdots \\
\Delta I_j \\
\Delta I_k \\
\vdots \\
0
\end{bmatrix} = \begin{bmatrix}
Z_{1j} \Delta I_j + Z_{1k} \Delta I_k \\
\vdots \\
Z_{jj} \Delta I_j + Z_{jk} \Delta I_k \\
Z_{kj} \Delta I_j + Z_{kk} \Delta I_k \\
\vdots \\
Z_{nj} \Delta I_j + Z_{nk} \Delta I_k
\end{bmatrix}
$$

(3.58)

We can therefore write the following two equations form (3.58)

$$
V_j = V_j^o + \Delta V_j = V_j^o + Z_{jj} \Delta I_j + Z_{jk} \Delta I_k
$$

$$
V_k = V_k^o + \Delta V_k = V_k^o + Z_{kj} \Delta I_j + Z_{kk} \Delta I_k
$$

The above two equations can be rewritten as

$$
V_j = V_j^o + (Z_{jj} - Z_{jk}) \Delta I_j + Z_{jk} (\Delta I_j + \Delta I_k)
$$

(3.59)

$$
V_k = V_k^o + Z_{kj} (\Delta I_j + \Delta I_k) + (Z_{kk} - Z_{kj}) \Delta I_k
$$

(3.60)

Since $Z_{jk} = Z_{kj}$, the network can be drawn as shown in Fig. 3.16. By inspection we can see that the open circuit voltage between the buses $k$ and $j$ is

$$
V_{oc,kj} = V_k^o - V_j^o
$$

(3.61)

and the short circuit current through these two buses is

$$
I_{sc,kj} = \Delta I_j = -\Delta I_k
$$

(3.62)

Also during the short circuit $V_k - V_j = 0$. Therefore combining (3.59) and (3.60) we get

$$
V_k - V_j = (V_k^o - V_j^o) + (2Z_{kj} - Z_{jj} - Z_{kk}) I_{sc,kj} = 0
$$

(3.63)

Combining (3.61) to (3.63) we find the Thevenin impedance between the buses $k$ and $j$ as

$$
Z_{th,kj} = \frac{V_{oc,kj}}{I_{sc,kj}} = Z_{jj} + Z_{kk} - 2Z_{kj}
$$

(3.64)

The above equation agrees with our earlier derivation of the two bus network given in (3.53).
Fig. 3.16 Thevenin equivalent between buses k and j.